B.Sc- part-I(H), paper-I Abs. Algebra 25-07-2020

Q.1. For any two elements a and b of a group G, show that G is abelian iff $(ab)^2 = a^2 b^2$

Soln: Let us first suppose that, G is abelian.

So that $ab = ba \ \forall a, b \in G$

Let us consider, $(ab)^2 = (ab)(ab) = a(ba)b$ [By associativity]

$= a(ab)b$ [By commutativity]

$= (aa)(bb)$ (By associativity)

$= a^2 b^2$

Thus, $(ab)^2 = a^2 b^2 ; \forall a, b \in G$.

Conversely: Let $(ab)^2 = a^2 b^2, \ \forall a, b \in G$

We have to show that $ab = ba$

Let us consider $(ab)^2 = a^2 b^2 \Rightarrow (ab)(ab) = (aa)(bb)$

$\Rightarrow a(ba)b = a(ab)b$ (By associativity)

$\Rightarrow ba = ab$ (By left and right cancellation laws)

Thus we have, $ab = ba, \forall a, b \in G$

Hence, G is abelian.

Q.2. Show that if G is an abelian group than for all $a, b \in G$ and all integers n

$$(ab)^n = a^n b^n$$

Soln: (I) Let $n = 0$

Then, $(ab)^0 = e$

Also, $a^0 b^0 = e.e = e$.

$\therefore (ab)^0 = a^0 b^0$

II Let $n > 0$

If $n = 1$, then $(ab)^1 = ab = a^1 b^1$

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Let us suppose our result is true for $n = r$

i.e. $(ab)^r = a^r b^r$

Then, $(ab)^{r+1} = (ab)^r \cdot ab = a^r b^r ab = a^r a b^r b$ $[\because ab^r = b^r a]$

$= a^{r+1} b^{r+1}$

Then, by mathematical induction for all $n > 0$,

$(ab)^n = a^n . b^n$

(iii) Let $n < 0$

Let $n = -r$, when r is a positive integer.

Then, $(ab)^n = (ab)^{-r} = [(ab)^r]^{-1} = [a^r b^r]^{-1} = [b^r a^r]^{-1}$

$[\because a^m b^m = b^m a^m]$

$= [a^r]^{-1} [b^r]^{-1}$ $[\because (ab)^{-1} = b^{-1} a^{-1}]$

$= a^{-r} b^{-r} = a^n b^n$

proved.

Q.3. If G is a group of even order, then show that there exist an element a, other than the identity 'e', such that $a^2 = e$

Soln: Let $o(G) = 2r$, where r is any positive integer.

Since, We know that, in a group every element possesses a unique inverse and $e^{-1} = e$. The remaining $(2r-1)$ elements should, therefore, we divided into pairs in such a way that each pair consists of two distinct element, which are inverse of each other But this is not possible, since $(2r-1)$ $(2r-1)$ is odd.

Hence, there exists an element $a \in G$, such that $a^{-1} = a$ where $a \neq e$.

But $a = a^{-1} \Rightarrow a^2 = a^{-1} a = e$

Thus, there exists $a \in G$ such that $a \neq e$ and $a^2 = e$

Solved.